Theoretical Computer Science 412 (2011) 2409–2417

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Hamiltonian properties of twisted hypercube-like networks with more faulty elements

Xiaofan Yang [a,c,*], Qiang Dong [a,b], Erjie Yang [a], Jianqiu Cao [c]

[a] College of Computer Science, Chongqing University, Chongqing 400044, China
[b] School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China
[c] School of Information Science and Engineering, Chongqing Jiaotong University, Chongqing 400074, China

ARTICLE INFO

Article history:
Received 2 September 2010
Received in revised form 12 December 2010
Accepted 30 January 2011
Communicated by R. Klasing

Keywords:
Interconnection networks
Fault tolerance
Hamiltonian cycle
Near Hamiltonian cycle
Twisted hypercube-like network

ABSTRACT

Twisted hypercube-like networks (THLNs) are a large class of network topologies, which subsume some well-known hypercube variants. This paper is concerned with the longest cycle in an n-dimensional (n-D) THLN with up to $2n-9$ faulty elements. Let G be an n-D THLN, $n \geq 7$. Let F be a subset of $V(G) \bigcup E(G)$, $|F| \leq 2n-9$. We prove that $G-F$ contains a Hamiltonian cycle if $\delta(G-F) \geq 2$, and $G-F$ contains a near Hamiltonian cycle if $\delta(G-F) \leq 1$. Our work extends some previously known results.

1. Introduction

Interconnection networks provide an effective mechanism for exchanging data in parallel and distributed computing systems. An interconnection network can be represented by a graph, where vertices and edges represent processors and communication links, respectively. The longest cycle in an interconnection network is an important issue because ring-structured parallel algorithms can be executed on such a cycle efficiently. Furthermore, the longest cycle problem should be studied in the presence of faulty elements, because with the increasing system size it becomes highly probable that there exist faults in a system.

Due to some appealing properties, hypercubes enjoy popularity as network topologies [16]. In order to further improve some specific properties of hypercubes, a number of hypercube variants, such as the crossed cubes [6], the twisted cubes [15], the Möbius cubes [2] and the locally twisted cubes [26], have been suggested. The fault-tolerant longest cycle embedding problems of these hypercube variants and other well-known interconnection networks have received considerable research attention [4,5,8,11–13].

The *hypercube-like networks* (HLNs, for short) are a large class of network topologies [1,7,17–21,25]. Among HLNs one may identify a subclass of networks, which in this paper are called the *twisted hypercube-like networks* (THLNs, for short). In particular, the above-mentioned hypercube variants are all THLNs. Park et al. [18] proved that all n-D THLNs with up to $n-2$ faulty elements possess Hamiltonian cycles.

A question arises naturally: What about the longest cycle in a THLN with more than $n-2$ faulty elements? This paper attempts to partially answer this question. Let G be an n-D THLN, $n \geq 7$. Let $F \subseteq V(G) \bigcup E(G)$, $|F| \leq 2n-9$. We will prove

* Corresponding author at: College of Computer Science, Chongqing University, Chongqing 400044, China. Tel.: +86 811 23 65342815.
E-mail addresses: xf_yang1964@yahoo.com, yxf640126@sina.com (X. Yang).

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.01.034

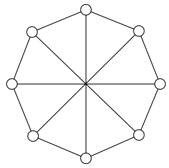

Fig. 1. 3D THLN.

that $G - F$ contains a Hamiltonian cycle if $\delta(G - F) \geq 2$, and $G - F$ contains a near Hamiltonian cycle if $\delta(G - F) \leq 1$. Our work extends some previously known results [13,23].

The rest of this paper is organized as follows. Section 2 gives preliminaries. Section 3 establishes the main result. Section 4 concludes the paper.

2. Preliminaries

For basic graph-theoretic notations and terminology, the reader is referred to Ref. [3]. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For two vertices u and v of a graph G, u is a *neighbor* of v if and only if $(u, v) \in E(G)$. For a vertex u of a graph G, let

$$N_G(u) = \{v \in V(G) : (u, v) \in E(G)\},$$
$$E_G(u) = \{(u, v) \in E(G) : v \in N_G(u)\},$$
$$\deg_G(u) = |N_G(u)|.$$

For a graph G and a set $F \subseteq V(G) \bigcup E(G)$, let $G - F$ denote a graph defined by

$$V(G - F) = V(G) - F,$$
$$E(G - F) = \{(u, v) \in E(G) : u, v \in V(G) - F \text{ and } (u, v) \notin F\}.$$

A path or a cycle in a graph is regarded as a subgraph of this graph. A *Hamiltonian cycle* (respectively, *Hamiltonian path*) in a graph is a cycle (respectively, path) that passes every vertex of the graph exactly once. A *near Hamiltonian cycle* (respectively, *near Hamiltonian path*) in a graph is a cycle (respectively, path) that passes every vertex but one of the graph exactly once. A path P starting from vertex u and terminating at vertex v is denoted $P[u, v]$. For a path P and two vertices x and y on P, the segment of P from x to y is denoted $P[x : y]$.

For two vertices u and v of a graph G, let $\mathrm{dist}_G(u, v)$ denote the distance between u and v, i.e., the minimum length of a path from u to v. For two vertices, u and v, on a cycle C (respectively, path P) of a graph, u is a *C-neighbor* (respectively, *P-neighbor*) of v if and only if $\mathrm{dist}_C(u, v) = 1$ (respectively, $\mathrm{dist}_P(u, v) = 1$). For a vertex u on path $P[w_1, w_2]$, a P-neighbor v of u is w_1*-closer* if and only if $\mathrm{dist}_P(v, w_1) < \mathrm{dist}_P(u, w_1)$.

For instance, consider a path $P = [w_1, w_2, w_3, w_4, w_5, w_6]$. Vertex w_3 has two P-neighbors: w_2 and w_4. Specifically, w_2 is the w_1-closer P-neighbor of w_3, and w_4 is the w_6-closer P-neighbor of w_3.

Definition 2.1. Let G be a graph.

(1) A *path cover* of G is a set of vertex-disjoint paths of G such that each vertex of G is passed by exactly one path. An *r-path cover* of G is a path cover of G containing r paths.

(2) A *near path cover* of G is a set of vertex-disjoint paths of G such that each vertex but one of G is passed by exactly one path. A *near r-path cover* of G is a near path cover of G that contains r paths.

(3) A path cover (respectively, near path cover) is *nontrivial* if every path in the cover contains at least two vertices. Otherwise this path cover (respectively, near path cover) is *trivial*.

According to this definition, a 1-path cover of a graph is essentially a Hamiltonian path of the graph, and a near 1-path cover of a graph is essentially a near Hamiltonian path of the graph.

Definition 2.2. For $n \geq 3$, an n-dimensional (n-D, for short) *twisted hypercube-like network* (THLN, for short) is a graph defined recursively as follows.

(1) A 3D THLN is isomorphic to the graph depicted in Fig. 1.

(2) For $n \geq 4$, an n-D THLN G is obtained from two vertex-disjoint $(n-1)$-D THLNs, G_1 and G_2, in this way:

$$V(G) = V(G_1) \bigcup V(G_2),$$
$$E(G) = E(G_1) \bigcup E(G_2) \bigcup \left\{\left(u, \phi(u)\right) : u \in V(G_1)\right\},$$

where $\phi : V(G_1) \to V(G_2)$ is a bijective mapping. In what follows, we will denote this graph G as $G = \oplus_\phi(G_1, G_2)$. Figs. 2–4 plot three typical 4D THLNs.

The following two lemmas are obviously true.

Lemma 2.1. *A THLN does not contain cycles of length* 3.

Lemma 2.2. *For any two distinct vertices, u and v, of a THLN G, we have*

$$\left|\left(N_G(u) \bigcup E_G(u)\right) \bigcap \left(N_G(v) \bigcup E_G(v)\right)\right| \leq 2.$$

The following two properties of THLNs will be used in the next section.

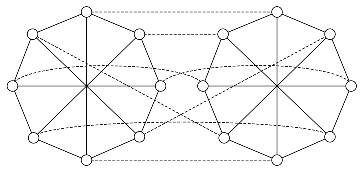

Fig. 2. 4D crossed cube.

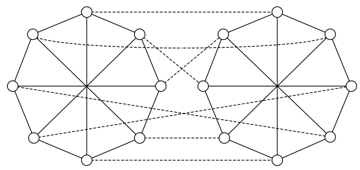

Fig. 3. 4D 0-Möbius cube.

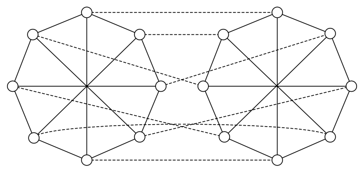

Fig. 4. 4D locally twisted cube.

Theorem 2.3. *[18] Let G be an n-D THLN,* $F \subseteq V(G) \bigcup E(G)$, $|F| \leq n-2$. *Then* $G - F$ *contains a Hamiltonian cycle.*

Theorem 2.4. *[19] Let G be an n-D THLN,* $F \subseteq V(G) \bigcup E(G)$. *Let* $u_1, v_1, \ldots, u_r, v_r$ *be* $2r$ *distinct vertices in* $V(G) - F$. *Suppose* $|F| + 2r \leq n - 1$. *Then* $G - F$ *contains an* r*-path cover of the form* $\{P_i[u_i, v_i] : 1 \leq i \leq r\}$.

3. Main result

The main result of this paper is presented as follows.

Theorem 3.1. *Let G be an n-D THLN,* $n \geq 7$. *Let* $F \subseteq V(G) \bigcup E(G)$, $|F| \leq 2n - 9$. *Then* $G - F$ *contains a Hamiltonian cycle if* $\delta(G - F) \geq 2$, *and* $G - F$ *contains a near Hamiltonian cycle if* $\delta(G - F) \leq 1$.

3.1. Useful lemmas

In order to prove the main theorem, we need to establish two lemmas.

Lemma 3.2. *Let G be an n-D THLN. Let* $F \subseteq V(G) \bigcup E(G)$, $|F| \leq 2n - 3$. *Suppose there is a vertex* $u \in V(G) - F$ *such that* $\deg_{G-F}(u) \leq 1$. *Then, for each vertex* $v \in V(G) - F - \{u\}$, *we have* $\deg_{G-F}(v) \geq 2n - 3 - |F|$.

Proof. In view of Lemma 2.2, we get

$$\begin{aligned}&\deg_{G-F}(v) \geq n - \left|F \bigcap \left(N_G(v) \bigcup E_G(v)\right)\right| \\ &\geq n - \left(|F| - \left|F \bigcap \left(N_G(u) \bigcup E_G(u)\right)\right| + \left|\left(N_G(u) \bigcup E_G(u)\right) \bigcap \left(N_G(v) \bigcup E_G(v)\right)\right|\right) \\ &\geq n - \left(|F| - (n-1) + 2\right) = 2n - 3 - |F|. \quad \square\end{aligned}$$

Lemma 3.3. *Let* G_1, G_2 *be two k-D THLNs,* $G = \oplus_\phi(G_1, G_2)$. *Let*

$$F \subseteq V(G) \bigcup E(G),\ F_1 = F \bigcap \left(V(G_1) \bigcup E(G_1)\right),\ F_2 = F \bigcap \left(V(G_2) \bigcup E(G_2)\right).$$

Let r be a positive integer such that $2r + |F_2| \leq k - 1$. *If* $G_1 - F_1$ *contains a nontrivial* r*-path cover (respectively, nontrivial near* r*-path cover),*

$$\left\{P_{1,1}[w_1, w_2], P_{1,2}[w_3, w_4], \ldots, P_{1,r}[w_{2r-1}, w_{2r}]\right\},$$

such that $\phi(w_i) \in N_{G-F}(w_i)$ *holds for* $1 \leq i \leq 2r$, *then* $G - F$ *contains a Hamiltonian cycle (respectively, near Hamiltonian cycle).*

Proof. By Theorem 2.4, $G_2 - F_2$ contains an r-path cover of the form

$$\left\{P_{2,1}[\phi(w_2), \phi(w_3)], P_{2,2}[\phi(w_4), \phi(w_5)], \ldots, P_{2,r}[\phi(w_{2r}), \phi(w_1)]\right\},$$

Then, $G - F$ contains the following Hamiltonian cycle (respectively, near Hamiltonian cycle):

$$\begin{aligned}&w_1 \xrightarrow{P_{1,1}} w_2 \xrightarrow{\{w_2,\phi(w_2)\}} \phi(w_2) \xrightarrow{P_{2,1}} \phi(w_3) \xrightarrow{\{\phi(w_3),w_3\}} w_3 \\ &\xrightarrow{P_{1,2}} w_4 \xrightarrow{\{w_4,\phi(w_4)\}} \phi(w_4) \xrightarrow{P_{2,2}} \phi(w_5) \xrightarrow{\{\phi(w_5),w_5\}} w_5 \\ &\rightarrow \cdots \\ &\xrightarrow{P_{1,r}} w_{2r} \xrightarrow{\{w_{2r},\phi(w_{2r})\}} \phi(w_{2r}) \xrightarrow{P_{2,r}} \phi(w_1) \xrightarrow{\{\phi(w_1),w_1\}} w_1. \quad \square\end{aligned}$$

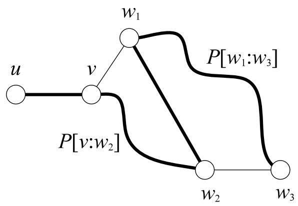

Fig. 5. A Hamiltonian path in $G_1 - F_1$.

3.2. Proof of the theorem

In this subsection, we complete the proof of Theorem 3.1. We argue the assertion by induction on n. Let G be a 7D THLN, $F \subseteq V(G) \bigcup E(G)$, $|F| \leqslant 2 \times 7 - 9 = 5 = 7 - 2$. By Theorem 2.3, $G - F$ contains a Hamiltonian cycle. Hence, the assertion is true for $n = 7$.

Suppose the assertion is true for $n = k \geq 7$. Let $G = \oplus_\phi(G_1, G_2)$ be a $(k+1)$-D THLN, where G_1, G_2 are k-D THLNs, $E_c = \{\{u, \phi(u)\} : u \in V(G_1)\}$. Let $F \subseteq V(G) \bigcup E(G)$, $|F| \leqslant 2(k+1) - 9 = 2k - 7$. Let

$F_1 = F \bigcap \big(V(G_1) \bigcup E(G_1)\big)$, $F_2 = F \bigcap \big(V(G_2) \bigcup E(G_2)\big)$, $F_c = F \bigcap E_c$.

Without loss of generality, we may assume $|F_1| \geq |F_2|$. Then, $|F_2| \leq k - 4$. There are three possibilities for the value of $|F_1|$, which will be treated, respectively.

Case 1. $|F_1| \leq 2k - 9$.

In this case, the assertion follows from the following two claims.

Claim 1. *If $\delta(G_1 - F_1) \geq 2$, then $G - F$ contains a Hamiltonian cycle.*

Proof. By the inductive hypothesis, $G_1 - F_1$ contains a Hamiltonian cycle C. Along C, there are at least $\lfloor (2^k - |F_1|)/2 \rfloor$ disjoint pairs of adjacent nodes. As

$$\left\lfloor \frac{2^k - |F_1|}{2} \right\rfloor \geq \frac{2^k - |F_1| - 1}{2} \geq 2^{k-1} - |F_1| - 1 \geq 2^{k-1} - 1 - [(2k-7) - |F_2| - |F_c|]$$
$$= |F_2| + |F_c| + (2^{k-1} - 2k + 6) > |F_2| + |F_c|,$$

$G_1 - F_1$ must contain a Hamiltonian path $P[w_1, w_2]$ such that $\phi(w_1) \in N_{G-F}(w_1)$ and $\phi(w_2) \in N_{G-F}(w_2)$. By Lemma 3.3 and in view of $|F_2| \leq k - 4$, we get that $G - F$ contains a Hamiltonian cycle. Claim 1 is proven. □

Claim 2. *Suppose $\delta(G_1 - F_1) \leq 1$. If $\delta(G - F) \leq 1$, then $G - F$ contains a near Hamiltonian cycle. Otherwise $G - F$ contains a Hamiltonian cycle.*

Proof. By the inductive hypothesis, $G_1 - F_1$ contains a near Hamiltonian cycle C. So, there is a unique vertex $u \in V(G_1) - F_1$ such that $u \notin V(C)$. Clearly, $\deg_{G_1 - F_1}(u) = \delta(G_1 - F_1) \leq 1$.

If $\delta(G - F) \leq 1$, like Claim 1 we can show that $G - F$ contains a near Hamiltonian cycle. Now, suppose $\delta(G - F) \geq 2$. Clearly, we have $\deg_{G_1 - F_1}(u) = 1$ and $\phi(u) \in N_{G-F}(u)$. Let $N_{G_1 - F_1}(u) = \{v\}$. Let w_1 be a C-neighbor of v. Let P be the path obtained by removing edge $\{v, w_1\}$ from C. By Lemma 3.2, we have

$\deg_{G_1 - F_1}(w_1) \geq 2k - 3 - |F_1| \geq 2k - 3 - [(2k-7) - |F_2| - |F_c|] = |F_2| + |F_c| + 4.$

This plus Lemma 2.1 implies that there is a vertex $w_2 \in N_{G_1 - F_1}(w_1) - u, v$ such that $\phi(w_3) \in N_{G-F}(w_3)$, where w_3 is the w_1-closer P-neighbor of w_2. Then $G_1 - F_1$ contains the following Hamiltonian path:

$u \xrightarrow{\{u,v\}} v \xrightarrow{P[v:w_2]} w_2 \xrightarrow{\{w_2,w_1\}} w_1 \xrightarrow{P[w_1:w_3]} w_3$ (see Fig. 5).

By Lemma 3.3, $G - F$ contains a Hamiltonian cycle. Claim 2 is proven. □

Case 2. $|F_1| = 2k - 8$.

Then, $|F_2| + |F_c| \leqslant 1$. Let $z \in F_1$. Then, $|F_1 - \{z\}| = 2k - 9$. One of the following two cases must occur.

Case 2.1. $\delta\,(G_1 - (F_1 - \{z\})) \geq 2$.

By the inductive hypothesis, $G_1 - (F_1 - \{z\})$ contains a Hamiltonian cycle. So, $G_1 - F_1$ contains a Hamiltonian path $P[w_1, w_2]$. In this case, the assertion follows from the following claim.

Claim 3. *If $\delta(G - F) \leq 1$, then $G - F$ contains a near Hamiltonian cycle. Otherwise $G - F$ contains a Hamiltonian cycle.*

Proof. First, suppose $\delta(G - F) \leq 1$. We may assume $\phi(w_1) \notin N_{G-F}(w_1)$. Let w_3 be the P-neighbor of w_1. As $P[w_3 : w_2]$ is a near Hamiltonian path of $G_1 - F_1$, it follows from Lemma 3.3 that $G - F$ contains a near Hamiltonian cycle.

Second, suppose $\delta(G - F) \geq 2$. If $\phi(w_1) \in N_{G-F}(w_1)$ and $\phi(w_2) \in N_{G-F}(w_2)$, it follows from Lemma 3.3 that $G - F$ contains a Hamiltonian cycle. Now, we may assume $\phi(w_1) \notin N_{G-F}(w_1)$. Then $\deg_{G_1 - F_1}(w_1) \geqslant 2$. Let w_3 be the P-neighbor of w_1. Then there is a vertex $w_4 \in N_{G_1 - F_1}(w_1) - \{w_3\}$. Let w_5 be the w_1-closer P-neighbor of w_4. Then $G_1 - F_1$ contains the following Hamiltonian path:

$w_5 \xrightarrow{P[w_5:w_3]} w_3 \xrightarrow{\{w_3,w_1\}} w_1 \xrightarrow{\{w_1,w_4\}} w_4 \xrightarrow{P[w_4:w_2]} w_2$ (see Fig. 6).

By Lemma 3.3, $G - F$ contains a Hamiltonian cycle. Claim 3 is proven. □

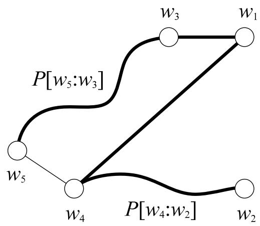

Fig. 6. A Hamiltonian path in $G_1 - F_1$.

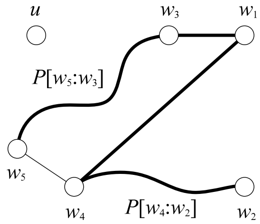

Fig. 7. A near Hamiltonian path in $G_1 - F_1$.

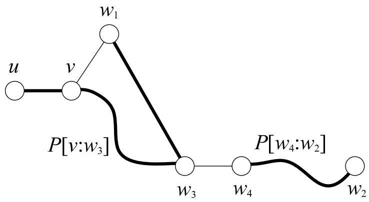

Fig. 8. A nontrivial 2-path cover in $G_1 - F_1$.

Case 2.2. $\delta\,(G_1 - (F_1 - \{z\})) \leq 1$.

By the inductive hypothesis, $G_1 - (F_1 - \{z\})$ contains a near Hamiltonian cycle C. So, there is a unique vertex $u \in V(G_1 - (F_1 - \{z\}))$ such that $u \notin V(C)$. Clearly, we have $\deg_{G_1-(F_1-\{z\})}(u) = \delta\,(G_1 - (F_1 - \{z\})) \leq 1$. In this case, the assertion follows from the following two claims.

Claim 4. *If $\delta(G - F) \leq 1$, then $G - F$ contains a near Hamiltonian cycle.*

Proof. Clearly, $C - \{z\}$ contains a near Hamiltonian path $P[w_1, w_2]$ of $G_1 - F_1$. If $\phi(w_1) \in N_{G-F}(w_1)$ and $\phi(w_2) \in N_{G-F}(w_2)$, it follows from Lemma 3.3 that $G - F$ contains a near Hamiltonian cycle. Now, assume $\phi(w_1) \notin N_{G-F}(w_1)$. Let w_3 be the P-neighbor of w_1. By Lemma 3.2, we get $\deg_{G_1-F_1}(w_1) \geq 5$. So, there is a vertex $w_4 \in N_{G_1-F_1}(w_1) - \{w_3\}$. Let w_5 be the w_1-closer P-neighbor of w_4. Then, $G_1 - F_1$ contains the following near Hamiltonian path:

$w_5 \xrightarrow{P[w_5:w_3]} w_3 \xrightarrow{\{w_3,w_1\}} w_1 \xrightarrow{\{w_1,w_4\}} w_4 \xrightarrow{P[w_4:w_2]} w_2$ (see Fig. 7).

By Lemma 3.3, $G - F$ contains a near Hamiltonian cycle. Claim 4 is proven. $\square$

Claim 5. *If $\delta(G - F) \geq 2$, then $G - F$ contains a Hamiltonian cycle.*

Proof. If $u = z$, then C is a Hamiltonian cycle of $G_1 - F_1$, and $G_1 - F_1$ contains a Hamiltonian path $P_1[w_1, w_2]$ such that $\phi(w_1) \in N_{G-F}(w_1)$ and $\phi(w_2) \in N_{G-F}(w_2)$. By Lemma 3.3, $G - F$ contains a Hamiltonian cycle. Now, suppose $u \neq z$. Then $C - \{z\}$ contains a near Hamiltonian path $P[w_1, w_2]$ of $G_1 - F_1$, $\deg_{G_1-F_1}(u) = 1$, and $\phi(u) \in N_{G-F}(u)$. Let $N_{G_1-F_1}(u) = \{v\}$. One of the following two cases must occur.

Case (i). $\phi(w_1) \in N_{G-F}(w_1)$ and $\phi(w_2) \in N_{G-F}(w_2)$. We distinguish three subcases.

Case (i)(a). Either $\mathrm{dist}_P(w_1, v) = 0$ or $\mathrm{dist}_P(v, w_2) = 0$. We may assume that the former is the case. Then $G_1 - F_1$ contains the following Hamiltonian path: $u \xrightarrow{\{u,v\}} v \xrightarrow{P} w_2$.

Case (i)(b). Either $\mathrm{dist}_P(w_1, v) = 1$ or $\mathrm{dist}_P(v, w_2) = 1$. We may assume that the former is the case. By Lemma 3.2, we have $\deg_{G_1-F_1}(w_1) \geqslant 5$. So, there is a vertex $w_3 \in N_{G-F}(w_1) - \{v\}$ such that $\mathrm{dist}_P(w_3, w_2) \geqslant 2$, and $\phi(w_4) \in N_{G-F}(w_4)$, where w_4 is the w_2-closer P-neighbor of w_3. Then $G_1 - F_1$ contains the following nontrivial 2-path cover:

$\left\{P[w_4 : w_2], u \xrightarrow{\{u,v\}} v \xrightarrow{P[v:w_3]} w_3 \xrightarrow{\{w_3,w_1\}} w_1\right\}$ (see Fig. 8).

Case (i)(c). $\mathrm{dist}_P(w_1, v) \geq 2$, $\mathrm{dist}_P(v, w_2) \geq 2$. Let w_3 be the w_1-closer P-neighbor of v. We may assume $\phi(w_3) \in N_{G-F}(w_3)$. Then $G_1 - F_1$ contains the following nontrivial 2-path cover:

$\left\{P[w_1 : w_3], u \xrightarrow{\{u,v\}} v \xrightarrow{P[v:w_2]} w_2\right\}$ (see Fig. 9).

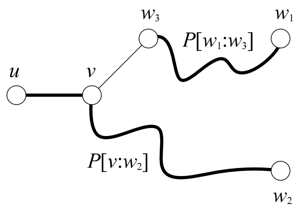

Fig. 9. A nontrivial 2-path cover in $G_1 - F_1$.

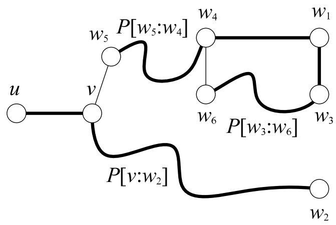

Fig. 10. A nontrivial 2-path cover in $G_1 - F_1$.

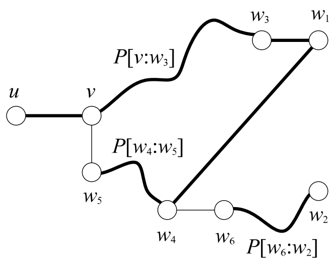

Fig. 11. A nontrivial 2-path cover in $G_1 - F_1$.

In either of the above three cases, it follows from Lemma 3.3 that $G - F$ contains a Hamiltonian cycle.

Case (ii). Either $\phi(w_1) \notin N_{G-F}(w_1)$ or $\phi(w_2) \notin N_{G-F}(w_2)$. We may assume that the former is the case. Let w_3 be the P-neighbor of w_1. By Lemma 3.2, we have $\deg_{G_1-F_1}(w_1) \geqslant 5$. So, one of the following two subcases must occur.

Case (ii)(a). On $P[w_1 : v]$ there is a vertex $w_4 \in N_{G_1-F_1}(w_1) - \{w_3, v\}$. Let w_5 be the w_1-closer P-neighbor of v, and let w_6 be the w_1-closer P-neighbor of w_4. Then $G_1 - F_1$ contains the following nontrivial 2-path cover:

$\left\{u \xrightarrow{\{u,v\}} v \xrightarrow{P[v:w_2]} w_2, w_5 \xrightarrow{P[w_5:w_4]} w_4 \xrightarrow{\{w_4,w_1\}} w_1 \xrightarrow{\{w_1,w_3\}} w_3 \xrightarrow{P[w_3:w_6]} w_6\right\}$ (see Fig. 10).

Case (ii)(b). On $P[v : w_2]$ there is a vertex $w_4 \in N_{G_1-F_1}(w_1) - \{w_3, v\}$ such that $\mathrm{dist}_P(w_4, w_2) \geqslant 2$. Let w_5 be the w_2-closer P-neighbor of v, and let w_6 be the w_2-closer P-neighbor of w_4. Then $G_1 - F_1$ contains the following nontrivial 2-path cover:

$\left\{P[w_6 : w_2], u \xrightarrow{\{u,v\}} v \xrightarrow{P[v:w_1]} w_1 \xrightarrow{\{w_1,w_4\}} w_4 \xrightarrow{P[w_4:w_5]} w_5\right\}$ (see Fig. 11).

In either of the above two cases, it follows from Lemma 3.3 that $G-F$ contains a Hamiltonian cycle. Claim 5 is proven. □

Case 3. $|F_1| = 2k - 7$. Then, $|F_2| = |F_c| = 0$. Let $z_1, z_2 \in F_1, z_1 \neq z_2$. Then, $|F_1 - \{z_1, z_2\}| = 2k - 9$. Now, we are faced with two possible cases.

Case 3.1. $\delta\,(G_1 - (F_1 - \{z_1, z_2\})) \geqslant 2$. By the inductive hypothesis, $G_1 - (F_1 - \{z_1, z_2\})$ contains a Hamiltonian cycle C. So, $C - \{z_1, z_2\}$ contains a 2-path cover of $G_1 - F_1$, $\{P_1[w_1, w_2], P_2[w_3, w_4]\}$, such that $\mathrm{length}(P_1) \leq \mathrm{length}(P_2)$. In this case the assertion follows from the following claim.

Claim 6. *If $\delta(G - F) \leq 1$, then $G - F$ contains a near Hamiltonian cycle. Otherwise $G - F$ contains a Hamiltonian cycle.*

Proof. First, suppose $\delta(G - F) \leq 1$. Then, $w_1 = w_2$, and P_2 is a near Hamiltonian path of $G_1 - F_1$. It follows from Lemma 3.3 that $G - F$ contains a near Hamiltonian cycle.

Second, suppose $\delta(G - F) \geq 2$. If $w_1 \neq w_2$, this claim follows directly from Lemma 3.3. Now, suppose $w_1 = w_2$. As $\deg_{G_1-F_1}(w_1) \geqslant 1$, there is a vertex $w_5 \in N_{G_1-F_1}(w_1)$. Without loss of generality, we assume $\mathrm{dist}_{P_2}(w_3, w_5) \geqslant 2$. Let w_6 be the w_3-closer P_2-neighbor of w_5. Then, $G_1 - F_1$ contains the following nontrivial 2-path cover:

$\left\{P_2[w_3 : w_6], w_1 \xrightarrow{\{w_1,w_5\}} w_5 \xrightarrow{P_2[w_5:w_4]} w_4\right\}$ (see Fig. 12).

It follows from Lemma 3.3 that $G - F$ contains a Hamiltonian cycle. Claim 6 is proven. □

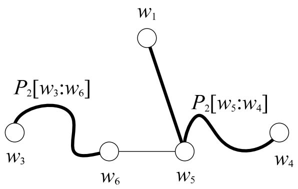

Fig. 12. A nontrivial 2-path cover in $G_1 - F_1$.

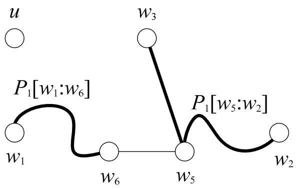

Fig. 13. A nontrivial near 2-path cover in $G_1 - F_1$.

Case 3.2. $\delta(G_1 - (F_1 - \{z_1, z_2\})) \leqslant 1$. By the inductive hypothesis, $G_1 - (F_1 - \{z_1, z_2\})$ contains a near Hamiltonian cycle C. So, there is a unique vertex $u \in V(G_1) - (F_1 - \{z_1, z_2\})$ such that $u \notin V(C)$ and
$\deg_{G_1-F_1}(u) \leqslant \deg_{G_1-(F_1-\{z_1,z_2\})}(u) = \delta(G_1 - (F_1 - \{z_1, z_2\})) \leqslant 1$.
In this case, the assertion is ensured by the following two claims.

Claim 7. *If $u \in \{z_1, z_2\}$, then $G - F$ contains a Hamiltonian cycle.*

Proof. We may assume $u = z_1$. Then, $C - \{z_2\}$ contains a Hamiltonian path of $G_1 - F_1$. By Lemma 3.3, $G - F$ contains a Hamiltonian cycle. Claim 7 is proven. □

Claim 8. *Suppose $u \notin \{z_1, z_2\}$. If $\delta(G - F) \leq 1$, Then $G - F$ contains a near Hamiltonian cycle. Otherwise $G - F$ contains a Hamiltonian cycle.*

Proof. $C - \{z_1, z_2\}$ contains a near 2-path cover $\{P_1[w_1, w_2], P_2[w_3, w_4]\}$ of $G_1 - F_1$ such that $\text{length}(P_1) \geq \text{length}(P_2)$.

First, suppose $\delta(G-F) \leq 1$. If $w_3 \neq w_4$, it follows from Lemma 3.3 that contains a near Hamiltonian cycle. Now, suppose $w_3 = w_4$. By Lemma 3.2, we have $\deg_{G_1-F_1}(w_3) \geqslant 4$. So there is a vertex $w_5 \in N_{G_1-F_1}(w_3) - \{u\}$. Let w_6 be the w_1-closer P_1-neighbor of w_5. Then, $G_1 - F_1$ contains the following nontrivial near 2-path cover:

$\left\{P_1[w_1 : w_6], w_3 \xrightarrow{\{w_3,w_5\}} w_5 \xrightarrow{P_1[w_5:w_2]} w_2\right\}$ (see Fig. 13).

It follows from Lemma 3.3 that $G - F$ contains a near Hamiltonian cycle.

Second, suppose $\delta(G - F) \geq 2$. Then $\deg_{G_1-F_1}(u) = 1$. Let $N_{G_1-F_1}(u) = \{v\}$. Without loss of generality, we assume $v \in V(P_1)$. Here there are totally four possible cases.

Case (i). $w_3 = w_4$. Let w_5 be the w_1-closer P_1-neighbor of v. By Lemma 3.2, we get $\deg_{G_1-F_1}(w_3) \geqslant 4$. Without loss of generality, we may assume that there is a vertex $w_6 \in N_{G_1-F_1}(w_3)$ such that w_6 lies on $P_1[w_1 : w_5]$ and $\text{dist}_{P_1}(w_1, w_6) \geqslant 2$. Let w_7 be the w_1-closer P_1-neighbor of w_6. Then, $G_1 - F_1$ contains the following nontrivial 3-path cover:

$\left\{P_1[w_1 : w_7], w_3 \xrightarrow{\{w_3,w_6\}} w_6 \xrightarrow{P_1[w_6,w_5]} w_5, u \xrightarrow{\{u,v\}} v \xrightarrow{P_1[v,w_2]} w_2\right\}$ (see Fig. 14).

Case (ii). $w_3 \neq w_4$, either $\text{dist}_{P_1}(w_1, v) = 0$ or $\text{dist}_{P_1}(v, w_2) = 0$, say, the former. Then, $G_1 - F_1$ contains the following nontrivial 2-path cover: $\left\{P_2, u \xrightarrow{\{u,v\}} v \xrightarrow{P_1} w_2\right\}$.

Case (iii). $w_3 \neq w_4$, either $\text{dist}_{P_1}(w_1, v) \geq 2$ or $\text{dist}_{P_1}(v, w_2) \geq 2$, say, the former is the case. Let w_5 be the w_1-closer P_1-neighbor of v. Then $G_1 - F_1$ contains the following nontrivial 3-path cover:

$\left\{P_2, P_1[w_1 : w_5], u \xrightarrow{\{u,v\}} v \xrightarrow{P_1[v:w_2]} w_2\right\}$ (see Fig. 15).

Case (iv). $w_3 \neq w_4$, $\text{dist}_{P_1}(w_1, v) = \text{dist}_{P_1}(v, w_2) = 1$, $w_1 \neq w_2$. By Lemma 3.2, we have $\deg_{G_1-F_1}(w_1) \geqslant 4$. So, there is a vertex $w_5 \in N_{G_1-F_1}(w_1)$ such that $\text{dist}_{P_2}(w_3, w_5) \geqslant 2$. Let w_6 be the w_3-closer P_1-neighbor of w_5. Then, $G_1 - F_1$ contains the following nontrivial 3-path cover:

$\left\{P_2[w_3 : w_6], u \xrightarrow{\{u,v\}} v \xrightarrow{\{v,w_2\}} w_2, w_1 \xrightarrow{\{w_1,w_5\}} w_5 \xrightarrow{P_2[w_5:w_4]} w_4\right\}$ (see Fig. 16).

In either of the above four cases, it follows from Lemma 3.3 that $G-F$ contains a Hamiltonian cycle. Claim 8 is proven. □

By combining the above discussions, we conclude that the assertion holds for $n = k + 1$. The inductive proof of this theorem is complete.

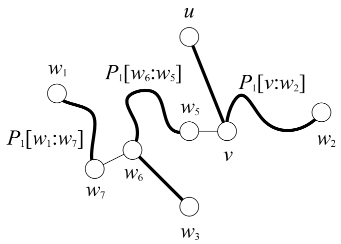

Fig. 14. A nontrivial 3-path cover in $G_1 - F_1$.

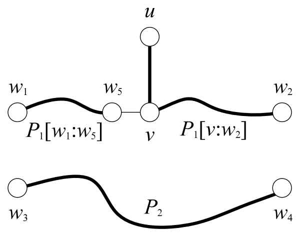

Fig. 15. A nontrivial near 3-path cover in $G_1 - F_1$.

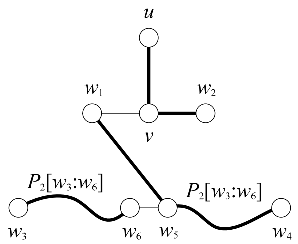

Fig. 16. A nontrivial 3-path cover in $G_1 - F_1$.

4. Concluding remarks

This paper has studied the longest cycle in an n-D THLN ($n \geq 7$) with a set F of up to $2n - 9$ faulty elements. We have proved that $G - F$ contains a Hamiltonian cycle if $\delta(G - F) \geq 2$, and $G - F$ contains a near Hamiltonian cycle if $\delta(G - F) \leq 1$. This result shows that THLNs enjoy excellent fault-tolerant Hamiltonian properties.

Park et al. [20] found that every n-D THLN G with fault set F contains a cycle of length l for any integer $4 \leq l \leq |V(G) - F|$, provided $|F| \leq n - 2$. For related work, see Refs. [9,10,14,22,24]. Inspired by this result and the work of this paper, we present the following conjecture.

Conjecture 4.1. *For $n \geq 7$, every n-D THLN G with fault set F contains a cycle of length l for any integer $4 \leq l \leq |V(G) - F| - 1$, provided $|F| \leq 2n - 9$.*

Acknowledgements

The authors would like to express their gratitude to the two anonymous reviewers for their valuable suggestions that greatly improved the quality of the paper. This work is supported by the National Nature Science Foundation of China (10771227).

References

[1] Y.C. Chen, J.J.M. Tan, L.H. Hsu, S.S. Kao, Superconnectivity and super-edge-connectivity for some interconnection networks, Appl. Math. Comput. 140 (2–3) (2003) 245–254.
[2] P. Cull, S.M. Larson, The Möbius cubes, IEEE Trans. Comput. 54 (2) (1995) 647–659.
[3] R. Diestel, Graph Theory, Springer-Verlag, New York, 2005.
[4] Q. Dong, X. Yang, Embedding a long fault-free cycle in a crossed cube with more faulty nodes, Inform. Process. Lett. 110 (11) (2010) 464–468.
[5] T. Dvorak, J. Fink, P. Gregor, V. Koubek, Long paths and cycles in faulty hypercubes: existence, optimality, complexity, Electron. Notes Discrete Math. 34 (1) (2009) 35–39.
[6] K. Efe, The crossed cube architecture for parallel computation, IEEE Trans. Parallel Distrib. Syst. 3 (5) (1992) 513–524.
[7] J. Fan, X. Lin, The t/k-diagnosability of the BC graphs, IEEE Trans. Comput. 54 (2) (2005) 176–184.
[8] J.S. Fu, Fault-free Hamiltonian cycles in twisted cubes with conditional link faults, Theoret. Comput. Sci. 407 (1–3) (2008) 318–329.
[9] S.Y. Hsieh, N.W. Chang, Hamiltonian path embedding and pancyclicity on the Möbius cube with faulty nodes and faulty edges, IEEE Trans. Comput. 55 (7) (2006) 854–863.

[10] S.Y. Hsieh, C.H. Chen, Pancyclicity on Möbius cubes with maximal edge faults, Parallel Comput. 30 (3) (2004) 407–421.
[11] S.Y. Hsieh, G.H. Chen, C.W. Ho, Hamiltonianlaceability of star graphs, Networks 36 (4) (2000) 225–232.
[12] S.Y. Hsieh, C.W. Ho, G.H. Chen, Fault-free Hamiltonian cycles in faulty arrangement graphs, IEEE Trans. Parallel Distrib. Syst. 10 (3) (1999) 223–237.
[13] S.Y. Hsieh, C.W. Lee, Conditional edge-fault hamiltonicity of matching composition networks, IEEE Trans. Parallel Distrib. Syst. 20 (4) (2009) 581–592.
[14] S.Y. Hsieh, C.W. Lee, Pancyclicity of restricted hypercube-like networks under the conditional fault model, SIAM J. Discrete Math. 23 (4) (2010) 2010–2019.
[15] P.A.J. Hilbers, M.R.J. Koopman, J.L.A. van de Snepscheut, The twisted cube, in: Lecture Notes in Computer Science, vol. 258, Parallel Architectures and Languages Europe, PARLE, Berlin, W. Germany, 1987, pp. 152–159.
[16] B. Parhami, An Introduction to Parallel Processing: Algorithms and Architectures, Plenum Press, New York, 1999.
[17] C.D. Park, K.Y. Chwa, Hamiltonian properties on the class of hypercube-like networks, Inform. Process. Lett. 91 (1) (2004) 11–17.
[18] J.H. Park, H.C. Kim, H.S. Lim, Fault-Hamiltonicity of hypercube-like interconnection networks, in: Proceedings of IEEE International Parallel & Distributed Processing Symposium, IPDPS, Denver, Colorado, 2005, pp. 60a–60a.
[19] J.H. Park, H.C. Kim, H.S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, IEEE Trans. Parallel Distrib. Syst. 17 (3) (2006) 227–240.
[20] J.H. Park, H.S. Lim, H.C. Kim, Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements, Theoret. Comput. Sci. 377 (1–3) (2007) 170–180.
[21] L.M. Shih, C.F. Chiang, L.H. Hsu, J.M. Tan, Strong Menger connectivity with conditional faults on the class of hypercube-like networks, Inform. Process. Lett. 106 (2) (2008) 64–69.
[22] H.L. Wang, J.W. Wang, J.M. Xu, Edge-fault-tolerant bipanconnectivity of hypercubes, Inform. Sci. 179 (4) (2009) 404–409.
[23] J.M. Xu, J.W. Wang, W.W. Wang, On super and restricted connectivity of some interconnection networks, Ars Combin. 94 (2010) 25–32.
[24] M.C. Yang, T.K. Li, J.M. Tan, L.H. Hsu, Fault-tolerant pancyclicity of the Möbius cubes, IEICE Trans. Fundam. Electron. Commun. Comput. Sci. E88-A (1) (2005) 346–352.
[25] X. Yang, J. Cao, G.M. Megson, J. Luo, Minimum neighborhood in a generalized cube, Inform. Process. Lett. 97 (3) (2006) 88–93.
[26] X. Yang, D.J. Evans, G.M. Megson, The locally twisted cubes, Int. J. Comput. Math. 82 (4) (2005) 401–413.